

# High codegree subgraphs in weakly quasirandom 3-graphs

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## Abstract

We show that we can extract large subgraphs with high minimum codegree from sequences of weakly quasirandom 3-graphs, for a particular notion of weakly quasirandom studied by Reiher, Rödl and Schacht.

In particular for any family of nonempty 3-graphs  $\mathcal{F}$ , the codegree density of  $\mathcal{F}$  is an upper bound on a certain weakly quasirandom Turán density for  $\mathcal{F}$ . This provides a partial answer to a question of Falgas-Ravry, Pikhurko, Vaughan and Volec.

## 1 Introduction

Given a set  $A$ , we write  $A^{(r)}$  for the collection of all  $r$ -tuples from  $A$  and  $[n]$  for the set  $\{1, 2, \dots, n\}$ . A 3-graph or *triple system* is a pair  $G = (V, E)$ , where  $V = V(G)$  is a set of vertices and  $E = E(G) \subseteq V^{(3)}$  is a collection of triples, which constitute the edges of  $G$ . We say  $G$  is *nonempty* if it contains at least one edge and set  $v(G) = |V(G)|$  and  $e(G) = |E(G)|$ . A *subgraph* of  $G$  is a 3-graph  $H$  with  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . The subgraph of  $G$  *induced* by a set  $X \subseteq V(G)$  is  $G[X] = (X, E(G) \cap X^{(3)})$ . We often identify  $G[X]$  with its vertex set  $X$  and write, for example  $e(X)$  for  $e(G[X])$ . We shall also usually write  $xy$  for the pair  $\{x, y\}$  and  $xyz$  for the triple  $\{x, y, z\}$ .

If  $G$  does not contain a copy of  $F$  as a subgraph, we say that  $G$  is  *$F$ -free*. The *Turán number*  $\text{ex}(n, F)$  of a nonempty 3-graph  $F$  is the maximum number of edges in an  $F$ -free 3-graph on  $n$  vertices, and its *Turán density* is the limit  $\pi(F) = \lim_{n \rightarrow \infty} \text{ex}(n, F) / \binom{n}{3}$  (this is easily shown to exist). In this paper we shall be interested in some variants Turán density.

The *codegree* of a pair  $xy \in V(G)^{(2)}$  is the number  $d(x, y)$  of edges of  $G$  containing the pair  $xy$ . The minimum codegree of  $G$ ,  $\delta_2(G)$ , is defined to be the minimum of  $d(x, y)$  over all pairs  $xy \in V(G)^{(2)}$ . The *codegree threshold*  $\text{ex}_2(n, F)$  of a nonempty 3-graph  $F$  is the maximum of  $\delta_2(G)$  over all  $F$ -free 3-graphs on  $n$  vertices. It can be shown [13] that the limit  $\pi_2(F) = \lim_{n \rightarrow \infty} \text{ex}_2(n, F) / (n-2)$  exists; this quantity is known as the *codegree density* of  $F$ . A simple averaging argument shows that

$$0 \leq \pi_2(F) \leq \pi(F) \leq 1,$$

and it is known that  $\pi_2(F) \neq \pi(F)$  in general.

## 1.1 Quasirandomness in 3-graphs

The study of quasirandom structures is a central, active and important area of research in extremal combinatorics, additive combinatorics and theoretical computer science, with many applications. The theory originates from the seminal papers of Thomason [20] and Chung–Graham–Wilson [1], who laid the ground for a rich and deep theory of quasirandomness for graphs. Two remarkable results of this theory are (i) that several notions of what it means for a graph to be ‘random-like’ turn out to be equivalent, and in particular (ii) that if a graph  $G$  on  $n$  vertices with  $p\binom{n}{2}$  edges, for some constant  $p \in (0, 1)$ , has *low discrepancy* — meaning  $\max_{X \subseteq V(G)} |e(G[X]) - p\binom{|X|}{2}| = o(n^2)$  — then for all graphs  $F$ , the number of copies of  $X$  in  $G$  is roughly what we would expect it to be if  $G$  was a genuinely random graph in which each edge is included independently at random with probability  $p$ :

$$\#\{\text{labelled copies of } F \text{ in } G\} = (1 + o(1))p^{e(F)}n^{v(F)}.$$

Since these foundational papers, there has been much interest in quasirandomness for hypergraphs, where the situation is quite different. Indeed, there is a host of non-equivalent notions of quasirandomness, and having low discrepancy in the sense above does *not* imply subgraph counts close to those you would expect in a random graph; as such it is only a ‘weak’ notion of quasirandomness. We refer a reader to the recent papers of Mubayi and Lenz [10] and Towsner [21] for a much more thorough presentation of current research on hypergraph quasirandomness.

The main motivation for this note comes from recent work of Reiher, Rödl and Schacht on extremal questions for quasirandom hypergraphs. These authors studied the following notions of weak quasirandomness for 3-graphs.

**Definition 1.1** ((1,1,1)-weak quasirandomness). A 3-graph  $G$  is  $(p, \varepsilon, (1, 1, 1))$ -weakly quasirandom if for every triple of set of vertices  $X, Y$  and  $Z \subseteq V$ , the number  $e_{1,1,1}(X, Y, Z)$  of triples  $(x, y, z) \in X \times Y \times Z$  such that  $xyz \in E(G)$  satisfies:

$$\left| e_{1,1,1}(X, Y, Z) - p|X||Y||Z| \right| \leq \varepsilon v(G)^3.$$

*Remark 1.2.* Using inclusion-exclusion, it is easy to show that the (1,1,1)-weakly quasirandom condition above is equivalent (up to a change in the error parameter  $\varepsilon$ ) to a low discrepancy condition of the form

$$\max_{X \subseteq V(G)} \left| e(G[X]) - p\binom{|X|}{3} \right| \leq \varepsilon' v(G)^3.$$

**Definition 1.3** ((1,2)-weak quasirandomness). A 3-graph  $G$  is  $(p, \varepsilon, (1, 2))$ -weakly quasirandom if for every set of vertices  $X \subseteq V$  and every set of pairs of vertices  $P \subseteq V^{(2)}$ , the number  $e_{1,2}(X, P)$  of pairs  $(x, p) \in X \times P$  such that  $\{x\} \cup p \in E(G)$  satisfies:

$$\left| e_{1,2}(X, P) - p|X||P| \right| \leq \varepsilon v(G)^3.$$

In the remainder of this paper, we shall write ‘wqr’ for ‘weakly quasirandom’.

**Definition 1.4** (wqr sequence). Let  $\star \in \{(1, 1, 1), (1, 2)\}$ . We say that a sequence  $\mathbf{G} = (G_n)_{n \in \mathbb{N}}$  of 3-graphs is  $\star$ -wqr if

- (i)  $v(G_n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and

- (ii) for each  $n$ ,  $G_n$  is  $(p, \varepsilon_n, \star)$ -wqr, for some constant  $p \in [0, 1]$  and a sequence of nonnegative reals  $\varepsilon_n$  tending to 0 as  $n \rightarrow \infty$ .

We refer to the quantity  $p$  as the *density* of the sequence  $\mathbf{G}$  and denote it by  $\rho(\mathbf{G})$ .

In a series of papers [15, 16, 17, 18], Reiher, Rödl and Schacht investigated the following problem.

**Problem 1.5.** Let  $F$  be a fixed nonempty 3-graph. For a given notion  $\star$  of weakly quasirandom, what is the least  $p \geq 0$  such that in every  $\star$ -wqr sequence  $\mathbf{G} = (G_n)$  with density strictly greater than  $p$ ,  $F$  is a subgraph of all but finitely many of the 3-graphs  $G_n$ ,  $n \in \mathbb{N}$ ?

When studying Problem 1.5, it is natural to define the following weakly-quasirandom analogue of Turán density.

**Definition 1.6** (wqr-Turán density). Let  $\star \in \{(1, 1, 1), (1, 2)\}$ , and let  $\mathcal{F}$  be a nonempty family of nonempty 3-graphs. The  $\star$ -wqr-Turán density of  $\mathcal{F}$  is defined to be

$$\pi_\star(\mathcal{F}) := \sup \left\{ \rho(\mathbf{G}) : \mathbf{G} \text{ is a } \star\text{-wqr sequence of } \mathcal{F}\text{-free 3-graphs} \right\}.$$

*Remark 1.7.* Clearly, if  $G$  is  $(p, \varepsilon, (1, 2))$ -wqr, then it is also  $(p, \varepsilon, (1, 1, 1))$ -wqr: given sets of vertices  $Y, Z$ , we can simply choose  $P = Y \times Z$  to be our set of pairs. This implies in particular that for any family  $\mathcal{F}$  as above,

$$0 \leq \pi_{(1,2)}(\mathcal{F}) \leq \pi_{(1,1,1)}(\mathcal{F}) \leq \pi(\mathcal{F}) \leq 1.$$

For  $\star \in \{(1, 1, 1), (1, 2)\}$ , Problem 1.5 thus asks to compute the  $\star$ -wqr Turán densities  $\pi_\star(F)$ . Note that other meaningful notions of weak quasirandomness for Problem 1.5 are possible see [10, 15, 21].

Problem 1.5 was first posed by Erdős and Sós [4] in the 1980s with  $\star = (1, 1, 1)$  for the complete 3-graph on 4 vertices  $K_4^{(3)}$ , and for the 3-graph  $K_4^{(3)-}$  obtained from  $K_4^{(3)}$  by deleting one of the edges. Settling one of their conjectures, Glebov, Král' and Volec [7] used flag-algebraic tools to prove that  $\pi_{(1,1,1)}(K_4^{(3)-}) = 1/4$ , with a lower bound construction due to Erdős and Hajnal [3] based on taking the oriented triangles in a random tournament. A second proof of their result using hypergraph regularity was later given by Reiher, Rödl and Schacht in [17]. Reiher, Rödl and Schacht further proved in [15] that  $\pi_{(1,2)}(K_4^{(3)}) = 1/2$ , with the lower bound coming from a random graph construction of Rödl [19]. It is a conjecture of Erdős that  $1/2$  remains the right value for the wqr-Turán density when  $(1, 2)$ -wqr is replaced by the weaker notion of  $(1, 1, 1)$ -wqr.

In a different direction, there has been much research on Turán densities for 3-graphs and, following work of Mubayi [12], on codegree densities. In the late 1990s, Nagle [14] and Nagle and Czygrinow [2] conjectured that  $\pi_2(K_4^{(3)-}) = 1/4$  and  $\pi_2(K_4^{(3)}) = 1/2$  respectively, with the same lower bound constructions as those for the  $(1, 1, 1)$ -wqr Turán density discussed above. Using flag-algebraic tools, Falgas-Ravry, Pikhurko, Vaughan and Volec [6] recently proved  $\pi_2(K_4^{(3)-}) = 1/4$ , settling the conjecture of Nagle, and showed all near-extremal constructions are close to the (set of) random tournament constructions of Erdős and Hajnal. They asked whether it was a coincidence that  $\pi_{(1,1,1)}(K_4^{(3)-}) \leq \pi_2(K_4^{(3)-})$ , and, more generally, whether one can extract high codegree sequences from sequences of weakly quasirandom 3-graphs.

Our contribution in this paper is to give a positive answer to this more general question, albeit with the stronger notion of  $(1, 2)$ -wqr replacing  $(1, 1, 1)$ -wqr (which in particular implies  $\pi_{(1,2)}(\mathcal{F}) \leq \pi_2(\mathcal{F})$ ). We should note that this is a somewhat surprising fact — indeed in Section 3 we construct

examples of  $(1,2)$ -wqr 3-graph with error parameter  $2\varepsilon$  in which no subgraph on more than  $1/\varepsilon$  vertices has non-zero codegree. So for a fixed  $\varepsilon$ , one cannot even have the size of a largest subgraph with non-zero codegree growing with the number of vertices!

Finally, we should remark that the extremal problems for codegree and for weak quasirandomness are very different in general. Let  $K_t^{(3)}$  denote the complete 3-graph on  $t$  vertices. As remarked in [5, 15], Rödl's construction can be adapted to give lower bounds for  $\pi_{(1,1,1)}(K_t^{(3)})$ ,  $\pi_{(1,2)}(K_t^{(3)})$  and  $\pi_2(K_t^{(3)})$  (and in the special case  $t = 6$  one can also give an alternative Ramsey-based random construction). However as shown in [5] the same lower bound for  $\pi_2(K_t^{(3)})$  can be attained by highly structured constructions. Furthermore, it follows from work of Lo and Markström [11] together with recent bounds on hypergraph Ramsey numbers due to Kostochka, Mubayi and Verstraëte [9] that for large  $t$  the bounds on  $\pi_2(K_t^{(3)})$  following from Rödl's constructions are superseded by blow-ups of certain random constructions, which fail to be  $(1,1,1)$ -wqr.

## 1.2 Our results

**Theorem 1.8.** *For any fixed  $\delta > 0$ , there exists  $m_0 > 0$  such that for every  $m \in \mathbb{N}$  with  $m \geq m_0$  there exists  $\varepsilon > 0$  such that any  $(p, \varepsilon, (1,2))$ -wqr 3-graph on  $n \geq m$  vertices contains an induced subgraph  $H$  on  $v(H) \geq m$  vertices whose minimum codegree satisfies*

$$\delta_2(H) \geq (p - \delta)v(H).$$

**Corollary 1.9.** *For any nonempty family of nonempty 3-graphs  $\mathcal{F}$ ,*

$$\pi_{(1,2)}(\mathcal{F}) \leq \pi_2(\mathcal{F}).$$

## 2 Finding high-codegree subgraphs in quasirandom hypergraphs

In this section we show how we can extract arbitrarily large subgraphs with high minimum codegree from sufficiently large  $(1,2)$ -wqr 3-graphs with sufficiently small error  $\varepsilon$ .

*Proof of Theorem 1.8.* We may assume without loss of generality that  $\delta > 0$  is small enough to ensure  $1080^2 \delta (\log \delta)^2 < 1$ , and  $512\delta^{-2} < 1080\delta^{-2} \log(1/\delta)$  as this only makes our task harder. Set  $m_0 = \lceil \frac{1080}{\delta^2} \log \frac{1}{\delta} \rceil$ , and fix  $m \geq m_0$ . Let  $n \geq m \geq m_0$  and  $\varepsilon = m^{-4}/4$ , so  $\varepsilon \leq \delta^2/4$ . Suppose  $G = (V, E)$  is a  $(p, \varepsilon, (1,2))$ -wqr 3-graph on  $n$  vertices. We claim that it contains an induced subgraph on  $m$  vertices with codegree at least  $(p - \delta)m$ . For  $p \leq \delta$ , we have nothing to prove, so we may assume that  $1 \geq p > \delta$ .

Call a pair  $xy \in V^{(2)}$  *poor* if  $d(x, y) < (p - \sqrt{\varepsilon})n$ , and *rich* otherwise. Let  $\mathcal{P}$  be the collection of all poor pairs. Then by the  $(1,2)$ -wqr assumption,

$$\sqrt{\varepsilon}n|\mathcal{P}| \leq |e_{1,2}(V, \mathcal{P}) - p|V||\mathcal{P}| \leq \varepsilon n^3.$$

Thus  $|\mathcal{P}| \leq \sqrt{\varepsilon}n^2$ . As each poor pair is contained in  $\binom{n-2}{m-2}$   $m$ -sets, it follows that there are at least

$$\binom{n}{m} - |\mathcal{P}| \binom{n-2}{m-2} > (1 - m^2\sqrt{\varepsilon}) \binom{n}{m} \geq \frac{1}{2} \binom{n}{m} \quad (2.1)$$

$m$ -sets of vertices which do not contain any poor pair.

Given a pair  $xy \in V^{(2)} \setminus \mathcal{P}$ , we call an  $m$ -subset  $S$  of  $V$  *bad* for  $xy$  if  $xy \subseteq S$  and  $|N(x, y) \cap S| \leq (p - \delta)m$ . Applying a standard Chernoff-type bound for the hypergeometric distribution (see e.g. [8]), we have

$$\begin{aligned} & \left| \left\{ S \in V^{(m)} : xy \subseteq S, |N(x, y) \cap S| \leq (p - \delta)m \right\} \right| \\ & \leq \left| \left\{ S' \in (V \setminus xy)^{(m)} : |N(x, y) \cap S'| \leq (p - 2\delta/3)(m - 2) \right\} \right| \\ & \leq \binom{n-2}{m-2} \exp \left( -\frac{(2\delta/3 - \sqrt{\varepsilon})^2(m-2)}{3(p - \sqrt{\varepsilon})} \right) \leq \binom{n-2}{m-2} \exp \left( -\frac{\delta^2 m}{216} \right), \end{aligned}$$

where the first inequality holds as  $m \geq 6/\delta$ , the application of the Chernoff bound gives us the second inequality and the last inequality holds since  $\varepsilon \leq \delta^2/4$  and  $m \geq 4$ .

An  $m$ -subset  $S$  of  $V$  is called *bad* if it is bad for some  $xy \in V^{(2)} \setminus \mathcal{P}$ . Hence, the number of bad  $m$ -subsets is at most

$$\begin{aligned} & \sum_{xy \in V^{(2)} \setminus \mathcal{P}} \left| \left\{ S \in V^{(m)} : xy \subseteq S, |N(x, y) \cap S| \leq (p - \delta)m \right\} \right| \\ & \leq \binom{n}{2} \binom{n-2}{m-2} \exp \left( -\frac{\delta^2 m}{216} \right) = \binom{n}{m} \binom{m}{2} \exp \left( -\frac{\delta^2 m}{216} \right) \\ & < \binom{n}{m} \frac{m_0^2}{2} \exp \left( -\frac{\delta^2 m_0}{216} \right) \leq \frac{1080^2 \delta \log^2 \delta}{2} \binom{n}{m} \leq \frac{1}{2} \binom{n}{m}, \end{aligned}$$

where the second inequality holds since for  $x > 512\delta^{-2}$  the function  $x \mapsto x^2 \exp \left( -\frac{\delta^2 x}{216} \right)$  is decreasing, and the last two inequalities hold by our choice of  $m_0$  and our assumption on  $\delta$ . Together with (2.1), this shows there exists an  $m$ -set inside which there is no poor pair and in which every rich pair has codegree at least  $(p - \delta)m$ . Such a set clearly gives us an induced subgraph of  $G$  on  $m$  vertices with minimum codegree at least  $(p - \delta)m$ .  $\square$

### 3 Quasirandom hypergraphs with no large high-codegree subgraphs

Keeping  $\delta$  fixed and considering a  $(1, 2)$ -wqr 3-graph with edge density  $p > \delta$ , the bounds obtained in the proof of Theorem 1.8 tells us we can find a subgraph on  $m = \Omega(\varepsilon^{-1/4})$  vertices with strictly positive codegree  $(p - \delta)m$ . With slightly more care, one can improve this to  $\Omega(\varepsilon^{-1/2})$ .

However, as we show below, we cannot guarantee the existence of any subgraph with strictly positive codegree on more than  $2/\varepsilon + 1$  vertices: our lower bound on  $m$  in Theorem 1.8 in terms of an inverse power of the error parameter  $\varepsilon$  is thus sharp up to the value of the exponent.

**Proposition 3.1.** *For every  $p \in (0, 1)$  and every  $\varepsilon > 0$ , there exists  $n_0$  such that for all  $n \geq n_0$  there exist  $(p, 2\varepsilon, (1, 2))$ -wqr 3-graphs in which every subgraph on  $m \geq \lfloor \varepsilon^{-1} \rfloor + 1$  vertices has minimum codegree equal to zero.*

*Proof.* Let  $G = (V, E)$  be a  $(p, \varepsilon, (1, 2))$ -wqr 3-graph on  $n$  vertices. Such a 3-graph can be obtained for example by taking a typical instance of an Erdős–Rényi random 3-graph with edge probability  $p$ . Consider a balanced partition of  $V$  into  $N = \lfloor \varepsilon^{-1} \rfloor$  sets  $V = \bigcup_{i=1}^N V_i$  with  $\lfloor n/N \rfloor \leq |V_1| \leq |V_2| \leq$

$\dots \leq |V_N| \leq \lceil n/N \rceil$ . Now let  $G'$  be the 3-graph obtained from  $G$  by deleting all triples that meet some  $V_i$  in at least two vertices for some  $i$ :  $1 \leq i \leq N$ .

By construction, every set of  $N + 1$  vertices in  $G'$  must contain at least two vertices from the same  $V_i$ , and thus must induce a subgraph of  $G$  with minimum codegree zero. Note that  $e(G) - e(G') \leq Nn \binom{\lceil n/N \rceil}{2} \leq n^3/N \leq \varepsilon n^3$ . Since  $G$  is  $(p, \varepsilon, (1, 2))$ -wqr, it follows that  $G'$  is  $(p, 2\varepsilon, (1, 2))$ -wqr.  $\square$

## 4 Concluding remarks

An obvious open question is whether we have

$$\pi_{(1,1,1)\text{-wqr}}(\mathcal{F}) \leq \pi_2(\mathcal{F}).$$

In other words: can always extract subgraphs with large minimum codegree from a  $(1, 1, 1)$ -wqr graph? Even obtaining large subgraphs with non-zero minimum codegree remains an open problem for this weaker notion of quasirandomness.

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